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INTEGRABLE SYSTEMS, OBTAINED BY POINT FUSION FROM RATIONAL AND ELLIPTIC GAUDIN SYSTEMS

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Using the procedure of the marked point fusion, there are obtained integrable systems with poles in the matrix of the Lax operator order higher than one, considered Hamiltonians, symplectic structure and symmetries of these systems. Also, taking the Inozemtsev Limit procedure it was found the Toda-like system having nontrivial commutative relations between the phase space variables.

1 Introduction

Rational and Elliptic Gaudin systems In the present work there were used Hamiltonians, Lax operators and symplectic structure of the rational and elliptic Gaudin systems. These systems were studied in the works [1] − [4]. Let us give their short description.

Let us begin from rational Gaudin system and consider n marked points $x_a, a = \overline{1,n}$ on CP^1 and assign a coadjoint orbit of the group $SL(N,C)$ to each point. The coordinates p_a^{ij} are functions on the corresponding orbits. The Lie-Poisson brackets of p_a^{ij} (spins) have the following form (Appendix B.):

$$\{p_a^{ij}, p_b^{kl}\} = \delta_{ab}(\delta^{il}p_a^{kj} - \delta^{kj}p_a^{il}) \tag{1}$$

On the phase space, which is the direct product of the orbits factorized by $SL(N,C)$ group with coordinate independent elements on CP^1, it is possible to define the integrable system. The Hamiltonians of this system commuting with respect to the Lie-Poisson brackets (1) have the following form:

$$H_a = \sum_{b\neq a} \frac{< p_a p_b >}{x_b - x_a}, \tag{2}$$

where $<>$ means the trace . Other Hamiltonians are also expressible in terms of the traces of the spin products and differences of coordinates at the marked points in certain degrees.

Let us describe now the elliptic case: instead of CP^1 let us consider the elliptic curve. The integrable system is determined on the symplectic factor-space R ([4]) which has the following form:

$$R = (u,v) \times (\prod_a \mathcal{O}_a // H), \tag{3}$$

where $\mathcal{O}_a$ - the coadjoint orbit of the $SL(N,C)$ group representation,
H - Cartan subgroup,
$u, v \in \mathcal{H}$ - Cartan subalgebra.

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The commutation relations between v,u and p_s^{ij} in addition to those between p_a^{ij} have the following form:

$$\{v^i, u^j\} = \delta^{ij}, \quad \{v^k, p_a^{ij}\} = 0, \quad \{u^k, p_a^{ij}\} = 0 \tag{4}$$

Well-known Hamiltonian of the elliptic Gaudin system describes a physical system consisting of n interacting particles:

$$H = \sum_i \frac{1}{2}(v^i)^2 + \sum_{i \neq j} < p^2 > \cdot E_2(u^{ij}), \tag{5}$$

where v_i - momenta,
$u^{ij} = u^i - u^j$ - difference of particle coordinates,
$E_2(u^{ij}) \equiv E_2(u^{ij}, \tau)$ - interaction potential of particles, represented by the elliptic Eisenstien functions. This function is connected with $\wp$-Weirstrass function: $E_2(z, \tau) = \wp(z, \tau) + 2\zeta(\frac{1}{2})$, where z - coordinate on the torus. Function $E_2(z, \tau)$ is expressed in terms of θ-function (Appendix L.): $E_2(z, \tau) = -\partial_z^2 \ln(\theta(z, \tau))$.

Note that in the previously mentioned works there are studied systems which are the examples of the Hitchin systems - integrable systems on the cotangent bundles to the moduli spaces of holomorphic bundles over Riemann curves. These systems may be obtained by the symplectic reduction ([4], [5]) from the input phase spaces. Using the Lax operator we get the important way to describe the integrable systems. So, in the case of rational Gaudin system the Lax operator takes the following form:

$$L(z) = \sum_a \frac{p_a}{(z - x_a)}, \tag{6}$$

where z - coordinate on CP^1.

The matrix of the Lax operator in the elliptic case takes the following form:

diagonal part -

$$L^{ii} = v^i + \sum_a p_a^{ii} \cdot E_1(z - x_a) \tag{7}$$

non-diagonal part -

$$L^{ij} = \sum_a p_a^{ij} \cdot \exp\left[-2\pi i u^{ij} \cdot \frac{(z - x_a) - (\bar{z} - \bar{x_a})}{\tau - \bar{\tau}}\right] \cdot \varphi(u^{ij}, z - x_a), \tag{8}$$

where:

$$\varphi(u^{ij}, z - x_a) = \frac{\theta(u^{ij} + z - x_a)\theta'(0)}{\theta(u^{ij})\theta(z - x_a)}. \tag{9}$$

In the case of the Hitchin system the matrix of the Lax operator coincides with the solution of the moment map equation appearing in the symplectic reduction procedure. The Hamiltonians of the integrable system come from the appropriate basis expansion. The Lax operators studied in the above-mentioned works admit the first order poles.

In the present work we try to answer the naturally arising questions: how do the systems with the higher poles in the Lax operator look like, what kind of symplectic form do these systems have and what are the moduli spaces of these systems.

Results and methods being used to obtain the systems We use the procedure of the point fusion to obtain the integrable systems from rational and elliptic Gaudin systems. The main idea of this method consists of finding such decomposition of the variables p, which gives us the pole order in the matrix of Lax operator being higher than one at some marked point, for example x_a, after taking the limit $x_b - x_a = \varepsilon \to 0$, where ε is the parameter of the point fusion(we put here the coefficient at ε equal to 1). It implies the existence of Hamiltonian and Casimir numbers in this new system being the same as in the initial one. This situation is realized by passing to the new variables p^r using grading decomposition:

$$p_a = \alpha_0 p^0 + \alpha_1 p^1 \varepsilon^{-1} + ... + \alpha_r p^r \varepsilon^{-r}, \tag{10}$$

where α_r - some variables.

The Hamiltonians in the case of two point (chosen from n marked point) fusion ($x_b \to x_a$) have the following form:

$$H_1^{2,2} = < (p^0)^2 > +2 \sum_{c \neq a,b} \frac{< p^1 p_c >}{(x_a - x_c)}, \quad H_1^{2,1} = \sum_{c \neq a,b}^{n} \left[\frac{< p^0 p_c >}{(x_a - x_c)} - \frac{< p^1 p_c >}{(x_a - x_c)^2} \right],$$
$$H_c^{2,1} = 2 \frac{< p^1 p_c >}{(x_c - x_a)^2} + \frac{< p^0 p_c >}{(x_c - x_a)} + \sum_{d \neq c,a,b} \frac{< p_c p_d >}{(x_c - x_d)}, \tag{11}$$

where $<>$ means the trace. The commutation relations in this case are determined in chapter 2, formula (22). The Lax operators of these systems were described in work [6]. Note, that in the case of r point (chosen from n marked point) fusion the Lax operator has the following form:

$$L_{new} = \frac{p^r \cdot k_r}{(z - x_a)^r} + ... + \frac{p^1 \cdot k_1}{(z - x_a)^2} + \frac{p^0 \cdot k_0}{(z - x_a)} + \sum_{c \neq a,b} \frac{p_c}{(z - x_c)}, \tag{12}$$

where k_r - some coefficients depending on the parameters of decomposition p^r, the parameters of the point fusion and some conditions arising from the requirement of the absence of singularities in the final expression. There is a correspondence between p^r and upper triangular matrices which are isomorphic to polynomial algebra of ε^{-1} variable with coefficients p^r. Commutation relations between p^r were found in chapter 2 (table (28)). Note, that special case of the Lax operator (12) was obtained in the work [7].

In the elliptic case the Hamiltonians of the two point fusion have the following form($p \in sl(2, C)$):

$$\begin{aligned} H^{2,2} = &-(p^0)^{12}(p^0)^{21} + E_1(u) \cdot [-(p^0)^{12}(p^1)^{21} + (p^0)^{21}(p^1)^{12}] + \\ &+(p^1)^{12}(p^1)^{21} \cdot [E_1^{'}(u) + E_1^2(u)] - 2v(p^1)^{11} \\ H^{2,0} = &\, v^2 + ((p^1)^{11})^2 \cdot [\frac{1}{6} E_1^{'''}(u) + (E_1^{'}(u))^2] + (p^0)^{12}(p^0)^{21} \cdot E_1^{'}(u) + \\ &+[(p^0)^{12}(p^1)^{21} - (p^0)^{21}(p^1)^{12}] \cdot [E_1(u)E_1^{'}(u) + \frac{1}{2} E_1^{''}(u)] - \\ &-(p^1)^{12}(p^1)^{21} \cdot [(E_1^{'}(u))^2 + E_1^2(u)E_1^{'}(u) + E_1(u)E_1^{''}(u) + \frac{1}{3} E_1^{'''}(u)], \end{aligned} \tag{13}$$

where $u, v \in \mathcal{H}$ belong to Cartan subalgebra,
$E_1(u^{ij}) \equiv E_1(u^{ij}, \tau)$ are the elliptic Eisenstien functions (Appendix A.) determined on the complex torus T^2 with modulus τ. This function is connected

with ζ-Weirstrass function: $E_1(z,\tau) = \zeta(z,\tau) + 2\zeta(\frac{1}{2})z$. $E_1(z,\tau)$ is expressed in terms of θ-function: $E_1(z,\tau) = \partial_z \ln(\theta(z,\tau))$.

In the elliptic case the matrix of the Lax operator of r point fusion has the following form:

diagonal part -

$$L^{ii} = v^i + (p^1)^{ii} \cdot \tilde{k}_2 \cdot E_1'(z - x_a) + ... + (p^r)^{ii} \cdot \tilde{k}_r \cdot E_1^{(r)}(z - x_a) \tag{14}$$

nondiagonal part -

$$L^{ij} = (p^0)^{ij} \cdot \tilde{k}_1 \cdot \varphi(u^{ij}, z - x_a) + ... + (p^r)^{ij} \cdot \tilde{k}_r \cdot \varphi^{(r)}(u^{ij}, z - x_a), \tag{15}$$

where $\tilde{k}_r$ is similar to k_r, commutation relations are the same as in the rational case. The Hamiltonians expressed by the variables invariant with respect to the gauge fixing are found in chapter 3, formula (50), and commutation relations - formula (53), (54).

The symplectic structure in the two point fusion case (rational and elliptic one) is described in chapter 2

The procedure called Inozemtsev Limit ([8]) is the method permits to obtain Toda-like system (the system with exponential type of the interaction) from the elliptic Gaudin system (the system with elliptic type of the interaction) by the combination of the trigonometric limit, the infinite shift of the particle coordinates and the renormalization of the interaction constant. The technical details are given in [8], [9]. The Hamiltonians of the Toda-like system , obtained by Inozemtsev limit from the elliptic Gaudin system in the case of the two point fusion and commutation relations were found in chapter 4, formulas (86),(84). Note, that the other Toda-like systems having nontrivial commutation relations between the phase space variables were studied in the work [10].

2 Systems arising from the rational Gaudin system

Let us begin from the rational Gaudin system. We will consider the two point fusion of n marked points ($n > 4$) and fulfil the following coordinate transformation and decomposition of the variables p:

$$\begin{gathered} x_b = x_a + \varepsilon \\ p_a = \alpha_0 p^0 + \alpha_1 p^1 \varepsilon^{-1}, \qquad p_b = \beta_0 p^0 + \beta_1 p^1 \varepsilon^{-1}, \end{gathered} \tag{16}$$

where α_r, β_r are some parameters. Putting (16) in the matrix of Lax operator (6) and taking the limit $\varepsilon \to 0$, we get:

$$L = -\frac{\alpha_0 p^0 + \alpha_1 p^1 \varepsilon^{-1}}{(z - x_a)} + \frac{\beta_0 p^0 + \beta_1 p^1 \varepsilon^{-1}}{(z - x_a - \varepsilon)} + \sum_{c \neq a,b} \frac{p_c}{(z - x_c)}. \tag{17}$$

During the calculation we need to put some additional conditions on parameters. The reason for their appearance is the requirement of the absence of singularities coming from degrees of ε^{-1} if $\varepsilon \to 0$. In the present case we have one condition $\alpha_1 + \beta_1 = 0$ and, setting $\alpha_0 = 1, \quad \alpha_1 = -1, \quad \beta_0 = 0, \quad \beta_1 = 1$, we get:

$$L_{new} = \frac{p^1}{(z - x_a)^2} + \frac{p^0}{(z - x_a)} + \sum_{c \neq a,b} \frac{p_c}{(z - x_c)} \tag{18}$$

The decomposition $< L^2_{new} >$ with respect to $(z-x_a)^{-m}$ has the following form :

$$< L^2_{new} >= H_1^{2,4}\cdot(z-x_a)^{-4}+H_1^{2,3}\cdot(z-x_a)^{-3}+H_1^{2,2}\cdot(z-x_a)^{-2}+H_1^{2,1}\cdot(z-x_a)^{-1}$$
$$+\sum_{c\neq a,b}(H_c^{2,2}\cdot(z-x_c)^{-2}+H_c^{2,1}\cdot(z-x_c)^{-1}) \tag{19}$$

where:

$$\begin{gathered}
H_1^{2,4} =< (p^1)^2 >, \quad H_1^{2,3}=2< p^1p^0 >, \\
H_1^{2,2} =< (p^0)^2 > +2\sum_{c\neq a,b}\frac{< p^1p_c >}{(x_a-x_c)}, \\
H_1^{2,1}=\sum_{c\neq a,b}\left[\frac{< p^0p_c >}{(x_a-x_c)}-\frac{< p^1p_c >}{(x_a-x_c)^2}\right], \quad H_c^{2,2} =< (p_c)^2 >, \\
H_c^{2,1}=2\frac{< p^1p_c >}{(x_c-x_a)^2}+\frac{< p^0p_c >}{(x_c-x_a)}+\sum_{d\neq c,a,b}\frac{< p_cp_d >}{(x_c-x_d)},
\end{gathered} \tag{20}$$

where $H_1^{2,4}, H_1^{2,3}, H_c^{2,2}$ are Casimir operators.

The condition $\sum\limits_a p_a=0$, which is the moment constraint corresponding to the action of the remnant gauge group, has to change because after the transformation of p_a and p_b, we get:

$$p^0+\sum_{c\neq a,b}p_c=0, \tag{21}$$

The numbers of Hamiltonians and Casimir operators do not change: instead of two Hamiltonians and Casimir operators of the rational Gaudin system related to the points x_a, x_b, there appeared $H_1^{2,4}, H_1^{2,3}$ and $H_1^{2,2}, H_1^{2,1}$, being Hamiltonians and Casimir operators correspondingly. It is possible to calculate the following commutation relations between the new variables:

$$\begin{gathered}
\{(p^1)^{ij},(p^1)^{kl}\}=0, \\
\{(p^0)^{ij},(p^0)^{kl}\}=\delta^{il}(p^0)^{kj}-\delta^{kj}(p^0)^{il}, \\
\{(p^0)^{ij},(p^1)^{kl}\}=\delta^{il}(p^1)^{kj}-\delta^{kj}(p^1)^{il},
\end{gathered} \tag{22}$$

It may be schematically presented as:

$$(p^1,p^1)\to 0, \quad (p^0,p^0)\to p^0, \quad (p^0,p^1)\to p^1, \tag{23}$$

Using the elements p^0, p^1, it is possible to compose matrixes of the form (in the case $r=2$):

$$P=\begin{bmatrix} p^0 & p^1 \\ 0 & p^0 \end{bmatrix} \tag{24}$$

, which generate the parabolic algebra. Taking into account the position of p in the matrix P (in the case when $p^r\in sl(2,C)$), it is possible to show that the commutation relations between p^r have the following form (Appendix B.):

$$\begin{gathered}
\{(p^0)^{ij},(p^1)^{kl}\}=\delta^{il}(p^1)^{k,j+N}-\delta^{kj}(p^1)^{i,l+N}, \\
\{(p^0)^{ij},(p^0)^{kl}\}=\frac{1}{2}[\delta^{il}((p^0)^{kj}+(p^0)^{k+N,j+N})-\delta^{kj}((p^0)^{il}+(p^0)^{i+N,l+N})],
\end{gathered} \tag{25}$$

where $N = 2$. The loop decomposition of the matrix P in the case of two point fusion gives polynomials of the form of $p^0 + p^1\varepsilon^{-1}$. The isomorphism can be established by comparing commutation relations for the matrix elements and for polynomials. Considering the transformation (16) for x and p in the case of fusion of the first r (for definiteness) of n marked points, we get the generalized expression for L_{new}, not depending on the sequence of those points of fusion but depending (up to the coefficients) on the number of points:

$$L_{new} = \frac{p^r \cdot k_r}{(z - x_a)^r} + ... + \frac{p^1 \cdot k_1}{(z - x_a)^2} + \frac{p^0 \cdot k_0}{(z - x_a)} + \sum_{c \neq a,b} \frac{p_c}{(z - x_c)} \tag{26}$$

The matrixes P, which generate the parabolic algebra, have the following form:

$$P = \begin{bmatrix} p^0 & p^1 & ... & p^{n-2} & p^{n-1} & p^n \\ 0 & p^0 & p^1 & ... & p^{n-2} & p^{n-1} \\ 0 & 0 & p^0 & p^1 & ... & p^{n-2} \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & ... & p^0 \end{bmatrix}. \tag{27}$$

Schematically, it is possible to present the commutation relations in the form of the table:

p	p^0	p^1	p^2	...	p^{n-1}	p^n
p^0	p^0	p^1	p^2	...	p^{n-1}	p^n
p^1	p^1	p^2	p^3	...	p^n	0
p^2	p^2	p^3	p^4	...	0	0
...	...	...	...	...	...	...
p^{n-1}	p^{n-1}	p^n	0	...	0	0
p^n	p^n	0	0	...	0	0

(28)

The intersection of the column and the row is the result of the commutation.

Symplectic form in the case of the point fusion Following works [2] – [4] let us consider the symplectic form of the Hitchin system in the case of n marked points on CP^1, selecting two points for the next fusion:

$$\Omega = \int_\Sigma \langle \delta A \wedge \delta \bar{A} \rangle + \sum_a \langle p_a, g_a^{-1} \delta g_a \rangle =$$

$$= \int_\Sigma [\langle \delta A \wedge \delta \bar{A} \rangle + \delta(z - x_a) \langle p_a, g_a^{-1} \delta g_a \rangle + \delta(z - x_b) \langle p_b, g_b^{-1} \delta g_b \rangle + \sum_{c \neq a,b} \delta(z - x_c) \langle p_c, g_c^{-1} \delta g_c \rangle], \tag{29}$$

where $<>$ is the Killing form, δ is the exterior differentiation operator,
-A and $\bar{A}$ are holomorphic and antiholomorphic parts of the connection correspondingly,
-$(p_a, g_a) \in T^*G_a$ where T^*G_a is the cotangent bundles at the marked point, $p_a \in Lie^*(G_a), g_a \in G_a$. Symplectic form is invariant with respect to the action of the group G_1:

$$G_1 = \{f(z, \bar{z}) \in C^\infty(\Sigma), G\} \tag{30}$$

The corresponding gauge transformations have the following form:

$$\begin{aligned} A &= fLf^{-1} + f\partial f^{-1}, \\ \bar{A} &= f\bar{L}f^{-1} + f\bar{\partial} f^{-1} \\ p_a &= f_a p_a f_a^{-1}, \quad g_a = g_a f_a^{-1}. \end{aligned} \tag{31}$$

The moment map equation takes the following form:

$$\bar{\partial}L = \delta(z - x_a)p_a + \delta(z - x_b)p_b + \sum_{c \neq a,b} \delta(z - x_c)p_c \tag{32}$$

After transformation (16), we get:

$$\bar{\partial}L = \delta(z - x_a)p^0 + \delta'(z - x_a)p^1 + \sum_{c \neq a,b} \delta(z - x_c)p_c. \tag{33}$$

Solving this equation, we get the Lax operator with the second order pole. Then let us find such transformation of the form Ω which gives us the new form Ω_{new} corresponding to (33). Let us introduce the logarithmic coordinates $\ln g$ on the group $Sl(N, C)$, which are the coordinates on the algebra $sl(N, C)$, then $\ln g_a \equiv X_a$, $\ln g_b \equiv X_b$ and $g_a^{-1}\delta g_a \equiv \delta X_a$, $g_b^{-1}\delta g_b \equiv \delta X_b$, and consider the following transformations:

$$X_a = X^0\varepsilon^{-1} - X^1, \quad X_b = -X^0\varepsilon^{-1} - X^1, \tag{34}$$

Let us define the coupling between the algebra X and the coalgebra P as $Res_{-1}Tr(PX)$. Substituting (16), putting $\alpha_0 = 1$, $\alpha_1 = -1$, $\beta_0 = 0$, $\beta_1 = 1$, and (34) to the form Ω, after expansion in ε, collecting similar terms and returning to $X^0 = (g^0)^{-1}\delta g^0$, we get the new form Ω_{new}:

$$\Omega = \int_\Sigma [\langle \delta A \wedge \delta \bar{A} \rangle + \delta(z - x_a)\langle p^0, g_0^{-1}\delta g_0 \rangle + \delta'(z - x_a)\langle p^1, g_0^{-1}\delta g_0 \rangle] + \sum_{c \neq a,b} \langle p_c, g_c^{-1}\delta g_c \rangle \tag{35}$$

It is possible to get (33) from this form.

3 Systems arising from the elliptic Gaudin system

Hamiltonians of Gaudin system in the case of two point fusion

Let us begin (as in the rational case) from the two point fusion and consider two marked points x_a and x_b. According to what was said in the introduction, the integrable system is defined on the symplectic factor-space R ([4]) which has the following form in the case of two points:

$$R = (u, v) \times (\mathcal{O}_a \times \mathcal{O}_b // H), \tag{36}$$

where $\mathcal{O}_a$ and $\mathcal{O}_b$ are the coadjoint orbits of the group $SL(N, C)$,
H is the Cartan subgroup,
$u, v \in \mathcal{H}$ belong to Cartan subalgebra.

The moment constraint corresponding to the action of the remnant gauge group has the following view:

$$\sum_a p^{ii} = 0 \tag{37}$$

Let us consider the case $N = 2$ (i.e. $p \in sl(2, C)$) and get the Hamiltonians by decomposing $< L^2 >$ into the sum of Eisenstein functions, where L is the matrix of the Lax operator in the holomorphic representation, with the diagonal part

$$L^{ii} = v^i + \sum_a p_a^{ii} \cdot E_1(z - x_a) \tag{38}$$

and non-diagonal part

$$L^{ij} = \sum_a p_a^{ij} \cdot \varphi(u^{ij}, z - x_a), \tag{39}$$

where:

$$\varphi(u^{ij}, z - x_a) = \frac{\theta(u^{ij} + z - x_a)\theta'(0)}{\theta(u^{ij})\theta(z - x_a)}. \tag{40}$$

In the case of the two point fusion we get:

$$\begin{gathered}
< L^2 >= (v^1 + p_a^{11} \cdot E_1(z - x_a) + p_b^{11} \cdot E_1(z - x_b))^2 + \\
+(v^2 + p_a^{22} \cdot E_1(z - x_a) + p_b^{22} \cdot E_1(z - x_b))^2 + \\
+p_a^{12} p_a^{21}(-E_1'(z - x_a) + E_1'(u^{12})) + p_b^{12} p_b^{21}(-E_1'(z - x_b) + E_1'(u^{12})) + \\
+2p_a^{12} p_b^{21} \cdot \varphi(u^{12}, z - x_a)\varphi(u^{21}, z - x_b) + 2p_a^{21} p_b^{12} \cdot \varphi(u^{21}, z - x_a)\varphi(u^{12}, z - x_b).
\end{gathered} \tag{41}$$

Analyzing zeroes and poles in $< L^2 >$ we get the decomposition in the following form (Appendix C.):

$$< L^2 >= \sum_a (E_1'(z - x_a) \cdot H_a^{2,2} + E_1(z - x_a) \cdot H_a^{2,1}) + H_a^{2,0} \tag{42}$$

, where:

$$H_a^{2,2} =< p_a^2 >, \tag{43}$$

$$\begin{gathered}
H_a^{2,1} = v^1 p_a^{11} + v^2 p_a^{22} + [p_a^{11} p_b^{11} + p_a^{22} p_b^{22}] \cdot E_1(x_a - x_b) + \\
+2p_a^{12} p_b^{21} \frac{\theta(u^{12} + x_b - x_a)\theta'(0)}{\theta(u^{12})\theta(x_a - x_b)} + 2p_a^{21} p_b^{12} \frac{\theta(u^{12} + x_a - x_b)\theta'(0)}{\theta(u^{12})\theta(x_b - x_a)},
\end{gathered} \tag{44}$$

$$\begin{gathered}
H^{2,0} = (v^1)^2 + (v^2)^2 - [p_a^{11} p_b^{11} + p_a^{22} p_b^{22}] \cdot (E_1'(x_a - x_b) + E_1^2(x_a - x_b)) + \\
+[p_a^{12} p_a^{21} + p_b^{12} p_b^{21}] \cdot E_1'(u^{12})) + \\
+2p_a^{12} p_b^{21}(E_1(u^{12}) - E_1(u^{12} + x_b - x_a)) \cdot \frac{\theta(u^{12} + x_b - x_a)\theta'(0)}{\theta(u^{12})\theta(x_a - x_b)} \\
+2p_a^{21} p_b^{12}(E_1(u^{12}) - E_1(u^{12} + x_a - x_b)) \cdot \frac{\theta(u^{12} + x_a - x_b)\theta'(0)}{\theta(u^{12})\theta(x_b - x_a)}.
\end{gathered} \tag{45}$$

Let us calculate the phase space dimension of the integrable system. There are eight variables: $u^{12}, v^1, p_a^{11}, p_a^{12}, p_a^{21}, p_b^{11}, p_b^{12}, p_b^{21}$ in the system which is in accordance with two Casimir operators and three Hamiltonians. Fixing the value of Casimir operators we choose the orbits $H_a^{2,2} = (\lambda_a)^2$, $H_b^{2,2} = (\lambda_b)^2$ and taking into account (37) we get:

$$(p_a^{11})^2 + p_a^{12} p_a^{21} = (\lambda_a)^2, \quad (p_b^{11})^2 + p_b^{12} p_b^{21} = (\lambda_b)^2, \quad p_a^{11} + p_b^{11} = 0 \tag{46}$$

Note, that the remnant gauge group in the general case (not necessarily in the holomorphic representation) consists of the doubly periodic functions of $(z, \bar{z})$ on the diagonal because they do not change the gauge fixing in the moment map equation (see Symmetries). Taking the Fourier-series expansion of these functions we get the basis in the space of the remnant gauge transformation. After that let us consider the gauge fixing being in accordance with

the coadjoint action of the remnant gauge group on p which are independent of $(z, \bar{z})$ diagonal matrices being the Cartan subgroup with the corresponding moment constraint (37). So we must consider functions which have only zero harmonic $c_0 = \exp(+\alpha)$ in the expansion and generate one-parameter class. The invariant with respect to the gauge fixing variables take the form:

$$p_a^{12}p_a^{21} \equiv x, \quad p_b^{12}p_b^{21} \equiv y, \quad p_a^{12}p_b^{21} \equiv z_1, \quad p_b^{12}p_a^{21} \equiv z_2, \quad p_a^{11}, \quad p_b^{11}, \quad u, \quad v \tag{47}$$

Finally, we get:

$$(p_a^{11})^2 + x = (\lambda_a)^2, \quad (p_b^{11})^2 + y = (\lambda_b)^2, \quad p_a^{11} + p_b^{11} = 0, \quad xy = z_1 z_2 \tag{48}$$

It is clear that six variables (all except u, v) can be expressed through two ones, for example z_1, z_2. So, we get four-dimensional phase space in accordance with two Hamiltonians. Note, that the Hamiltonians, as functions on this phase space, are invariant with respect to the action of the remnant gauge subgroup because phase space is the factor-space with respect to the action of this subgroup.

Hamiltonians of the system obtained by the two point fusion
Let us realize the point fusion in accordance with the transformation (16). Note, that commutation relations between $(p^0)^{ij}$ and $(p^1)^{ij}$ do not depend on the elliptic function and are the same as in the rational case. The additional issue in the evaluation of these limits is the decomposition of the elliptic function into series of ε. In the case of $p \in sl(2, C)$ after calculation, setting $u^{12} \equiv u$ and $v^1 = -v^2 = v$, we get (Appendix D.):

$$\frac{1}{2} < L_{new}^2 >= E_1^{'''}(z - x_a) \cdot H^{2,4} + E_1^{''}(z - x_a) \cdot H^{2,3} + E_1^{'}(z - x_a) \cdot H^{2,2} + H^{2,0}, \tag{49}$$

Where :

$$\begin{gathered} H^{2,4} = -\frac{1}{6} \cdot \frac{bc}{a} - \frac{1}{6} \cdot g^2 = -\frac{1}{12} < (p^1)^2 >, \quad H^{2,3} = \frac{1}{2} \cdot (b + c) = \frac{1}{2} < (p^0 p^1) >, \\ H^{2,2} = -a + (c - b) \cdot E_1(u) + \frac{bc}{a} \cdot [E_1^{'}(u) + E_1(u)^2] - 2vg, \\ H^{2,0} = v^2 + g^2 \cdot [\frac{1}{6} E_1^{'''}(u) + (E_1^{'}(u))^2] + a \cdot E_1^{'}(u) + (b - c) \cdot [\frac{1}{2} E_1^{''}(u) + E_1^{'}(u) \cdot E_1(u)] - \\ -\frac{bc}{a} \cdot [\frac{1}{3} E_1^{'''}(u) + (E_1^{'}(u))^2 + E_1^{''}(u) \cdot E_1(u) + E_1^{'}(u) \cdot E_1(u)^2], \end{gathered} \tag{50}$$

where we use the following notation for the variables invariant with respect to the gauge fixing:

$$(p^0)^{12}(p^0)^{21} = a, \quad (p^0)^{12}(p^1)^{21} = b, \quad (p^1)^{12}(p^0)^{21} = c, \quad (p^1)^{12}(p^1)^{21} = d = \frac{bc}{a}, \quad (p^1)^{11} = g \tag{51}$$

Let us calculate the dimension of the phase space of the new system. There are eight variables in the space $(u, v) \times (p^r)^{ij}$. $H^{2,4}$ and $H^{2,3}$ are the Casimir operators. The moment map for the remnant gauge action takes the form $(p^0)^{11} = 0$. So, we get the four-dimensional phase space and two Hamiltonians. Fixing the Casimir operators gives the equations:

$$\frac{bc}{a} + g^2 = \lambda_0, \quad b + c = \lambda_1 \tag{52}$$

So we get four dimensional phase space and two Hamiltonians. The Poisson brackets in the new variables take the following form:

$$\begin{gathered}\{v,u\}=1,\quad \{c,b\}=0,\quad \{a,g\}=c-b,\\ \{c,a\}=-2ag,\quad \{b,a\}=2ag,\quad \{b,g\}=-\frac{bc}{a},\quad \{c,g\}=\frac{bc}{a}.\end{gathered}\tag{53}$$

The Poisson brackets for $d\equiv\frac{bc}{a}$ take the following form:

$$\{\frac{bc}{a},a\}=2g(b-c),\quad \{\frac{bc}{a},g\}=0,\quad \{\frac{bc}{a},b\}=-2g\frac{bc}{a},\quad \{\frac{bc}{a},c\}=2g\frac{bc}{a}\tag{54}$$

Symmetries Hamiltonians defined on the phase space must be invariant with respect to the remnant gauge action - the Bernstein-Schvartsman group, which is the semi-direct product of the Weyl group and the lattice shifts.

Let us consider (in accordance with [4]) the gauge transformations preserving the gauge fixing $\bar{L}=diag(sl[2,C])$, chosen in the moment map equation:

$$\bar{\partial}L+\frac{2\pi i}{\tau-\bar{\tau}}\cdot[\bar{L},L]=2\pi i\cdot(\delta^2(x_a)p_a+\delta^2(x_b)p_b),\tag{55}$$

with boundary conditions

$$L(z+1)=L(z),\qquad L(z+\tau)=L(z),\tag{56}$$

where:

$$\bar{L}=\begin{bmatrix}u^1 & 0\\ 0 & u^2\end{bmatrix}.\tag{57}$$

Then the remnant gauge group in the general case consists of the doubly periodic Cartan valued functions of $(z,\bar{z})$ because they preserve the gauge fixing in the moment map equation. Taking the Fourier-series expansion of these functions we get basis in the space of the remnant gauge transformation which consist of harmonic having the following form:

$$f^i=\exp\left[2\pi i\cdot(m^i\frac{z-\bar{z}}{\tau-\bar{\tau}}+n^i\frac{\tau\bar{z}-\bar{\tau}z}{\tau-\bar{\tau}})\right],\quad m^i,n^i\in Z\tag{58}$$

Let us consider the transformation of $\bar{L}$ under the basis function action:

$$2\pi i\frac{1}{\tau-\bar{\tau}}\begin{bmatrix}u^1 & 0\\ 0 & u^2\end{bmatrix}\to f\cdot 2\pi i\frac{1}{\tau-\bar{\tau}}\begin{bmatrix}u^1 & 0\\ 0 & u^2\end{bmatrix}\cdot f^{-1}+f\bar{\partial}f^{-1},\tag{59}$$

where:

$$f=\begin{bmatrix}f^1 & 0\\ 0 & f^2\end{bmatrix},\tag{60}$$

From this we get law of the transformation for u^i:

$$u^i\to u^i+m^i-n^i\tau\tag{61}$$

Let us consider the transformation of u^{12} in the following form $u^{12}\to u^{12}+1=u^1-u^2+1$. We may represent this transformation as the following two: $u^1\to u^1+1$ and $u^2\to u^2$. Then f take the form $(m^1=1,n^1=0)$:

$$f=\begin{bmatrix}f^1 & 0\\ 0 & 1\end{bmatrix},\quad f^1=\exp\left[2\pi i\cdot\frac{z-\bar{z}}{\tau-\bar{\tau}}\right],\tag{62}$$

It is possible to define the action of f on p_a:

$$p_a \to f_a p_a f_a^{-1} = \begin{bmatrix} p_a^{11} & p_a^{12} \cdot f_a^1 \\ p_a^{21} \cdot (f_a^1)^{-1} & p_a^{22} \end{bmatrix} \tag{63}$$

We obtain the transformation law for p_a^{ij}:

$$p_a^{ii} \to p_a^{ii}, \quad p_a^{12} \to p_a^{12} \exp\left[2\pi i \cdot \frac{x_a - \bar{x_a}}{\tau - \bar{\tau}}\right], \quad p_a^{21} \to p_a^{21} \exp\left[-2\pi i \cdot \frac{x_a - \bar{x_a}}{\tau - \bar{\tau}}\right] \tag{64}$$

It is possible to obtain similar expressions for $u^1 \to u^1 + \tau$, in the general case we have:

$$\begin{aligned} u^{ij} &\to u^{ij} + m^i - n^i \tau \\ p_a^{ii} &\to p_a^{ii}, \\ p_s^{ij} &\to p_s^{ij} \cdot \exp\left[2\pi i \cdot m^i \frac{x_a - \bar{x_a}}{\tau - \bar{\tau}} + 2\pi i \cdot n^i \frac{\bar{\tau} x_a - \bar{x_a}\tau}{\tau - \bar{\tau}}\right], \\ \varphi(u^{ij}, z - x_a) &\to \varphi(u^{ij}, z - x_a) \cdot \exp\left[+2\pi i n^i \cdot (z - x_a)\right]. \end{aligned} \tag{65}$$

Transformations of the system obtained by point fusion Let us do the gauge transformations of $L, \bar{L}, p_s^{ij}$ in the moment map equation using the following function:

$$\tilde{f} = \begin{bmatrix} \exp\left[-2\pi i u^1 \cdot \dfrac{z - \bar{z}}{\tau - \bar{\tau}}\right] & 0 \\ 0 & \exp\left[-2\pi i u^2 \cdot \dfrac{z - \bar{z}}{\tau - \bar{\tau}}\right] \end{bmatrix}, \tag{66}$$

So we get the moment map equation in the following form:

$$\bar{\partial} L = 2\pi i \cdot (\delta^2(x_a) p_a + \delta^2(x_b) p_b), \tag{67}$$

with the boundary conditions for the matrix of the Lax operator:

$$L^{ij}(z+1) = L^{ij}(z), \qquad L^{ij}(z+\tau) = L^{ij}(z) \cdot \exp\left[-2\pi i u^{ij}\right] \tag{68}$$

Nondiagonal part of the matrix of the Lax operator take the holomorphic form:

$$L^{ij} = p_a^{ij} \cdot \frac{\theta(u^{ij} + z - x_a)\theta'(0)}{\theta(u^{ij})\theta(z - x_a)} + p_b^{ij} \cdot \frac{\theta(u^{ij} + z - x_b)\theta'(0)}{\theta(u^{ij})\theta(z - x_b)} \tag{69}$$

It is necessary to express the new variables through old ones $(p_a)^{ij}_{new} = p_a^{ij} \cdot \tilde{f}^{ij} (\tilde{f}^{ji})^{-1}$ to find the transformations for the new variables $(p_a)^{ij}_{new}$ in accordance to the transformations $u^{ij} \to u^{ij} + m^i - n^i \tau$. Having found the eventual transformation we get it in the holomorphic gauge:

$$\begin{aligned} u^{ij} &\to u^{ij} + m^i - n^i \tau \\ p_s^{ij} &\to p_s^{ij} \cdot \exp\left[-2\pi i \cdot n^i x_s\right] \end{aligned} \tag{70}$$

Now we can consider the transformations of the system obtained by point fusion. Having done the overscaling transformations of p_a^{ij} in the matrix of the Lax operator in the holomorphic gauge and at the same time the transformations of the variables under the shift u^{ij}, we get the modified action of

the Bernstein-Schvartsman group on the new variables $(p^r)^{ij}$, $\varphi(u^{ij}, z - x_a)$ and its derivative, having the following form:

$$u^{ij} \equiv u \to u + m - n\tau,$$
$$(p^0)^{ij} \to (p^0)^{ij} - 2\pi in \cdot (p^1)^{ij}, \qquad (p^1)^{ij} \to (p^1)^{ij}, \qquad (p^1)^{ii} \to (p^1)^{ii},$$
$$\varphi(u, z - x_a) \to \varphi(u, z - x_a) \cdot \exp\left[+2\pi in \cdot (z - x_a)\right],$$
$$\varphi^{'}(u, z - x_a) \to \left[-2\pi in \cdot \varphi(u, z - x_a) + \varphi^{'}(u, z - x_a)\right] \cdot \exp\left[+2\pi in \cdot (z - x_a)\right] \tag{71}$$

Matrix of the Lax operator of the new system Let us consider the matrix of the Lax operator in the holomorphic gauge in the case of two points . Having done transformations (16) and the decomposition in ε of the resulting expression, passing to the limit $\varepsilon \to 0$ we get:

diagonal part -

$$L^{ii} = v^i - (p^1)^{ii} \cdot E_1^{'}(z - x_a) \tag{72}$$

nondiagonal part -

$$L^{ij} = (p^0)^{ij} \cdot \varphi(u^{ij}, z - x_a) - (p^1)^{ij} \cdot \varphi^{'}(u^{ij}, z - x_a) \tag{73}$$

The matrix of the Lax operator and the corresponding Hamiltonians depend on the method of the decomposition of p^r. It means that it is possible to consider decomposition in ε and $\bar{\varepsilon}$. In the present work we consider holomorphic decomposition p^r parameterized by ε.

In the elliptic case with two marked points the total moduli space of the system is a fibered space with the base defined by the elliptic module τ. The fiber of this space is a torus with marked point. This marked point corresponds to a new system having there a pole of second order in the matrix of Lax operator. This follows from the considering the module $x_b - x_a$ (with fixed τ). In the other words we can fix, for example x_a and then x_b will take value in the torus. When $x_b \to x_a$ the new system obtained by the fusion of x_b and x_a corresponds to the fixed point.

Generalizing to the case of r point fusion we get the matrix of the Lax operator of r point fusion in the following form:

diagonal part -

$$L^{ii} = v^i + (p^1)^{ii} \cdot \tilde{k}_2 \cdot E_1^{'}(z - x_a) + ... + (p^r)^{ii} \cdot \tilde{k}_r \cdot E_1^{(r)}(z - x_a) \tag{74}$$

nondiagonal part -

$$L^{ij} = (p^0)^{ij} \cdot \tilde{k}_1 \cdot \varphi(u^{ij}, z - x_a) + ... + (p^r)^{ij} \cdot \tilde{k}_r \cdot \varphi^{(r)}(u^{ij}, z - x_a), \tag{75}$$

where $\tilde{k}_r$ is similar to k_r, commutation relations are the same as in the rational case. Note, that the symplectic structure is the same as in the rational case, too. Integration in the formula (35) is taken over the torus.

4 Inozemtsev Limit

In the Inozemtsev Limit procedure we must satisfy the requirement of the absence of singularities in the integrals of motion which are the traces of the Lax operators which coincide with the solution of the moment map equation:

$$\frac{1}{2} < L^2 >, \ \frac{1}{3} < L^3 >, \ ..., \frac{1}{k} < L^k > \tag{76}$$

Taking the Eisenstein-series expansion we get required number of Hamiltonians as the coefficients of the decomposition. Let us write the first nonvanishing term of the Eisenstein functions in the limit $\omega_2 \to \infty$, taking into account $u = \tilde{u} + t\omega_2$, where t is some parameter:

function	$t = 0$	$0 < t < 1$	$t = 1$	$1 < t < 2$
$E_1(u)$	$+\frac{1}{2}\frac{\cosh(\frac{\tilde{u}}{2})}{\sinh(\frac{\tilde{u}}{2})}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$
$E_1^{'}(u)$	$-\frac{1}{4}\frac{1}{\sinh^2(\frac{\tilde{u}}{2})}$	$-e^{-\tilde{u}-t\omega_2}$	$-2\cosh(\tilde{u})\cdot e^{-\omega_2}$	$-e^{+\tilde{u}-(2-t)\omega_2}$
$E_1^{''}(u)$	$+\frac{1}{4}\frac{\cosh(\frac{\tilde{u}}{2})}{\sinh^3(\frac{\tilde{u}}{2})}$	$+e^{-\tilde{u}-t\omega_2}$	$-2\sinh(\tilde{u})\cdot e^{-\omega_2}$	$-e^{+\tilde{u}-(2-t)\omega_2}$
$E_1^{'''}(u)$	$-\frac{1}{4}\frac{1}{\sinh^2(\frac{\tilde{u}}{2})} - \frac{3}{8}\frac{1}{\sinh^4(\frac{\tilde{u}}{2})}$	$-e^{-\tilde{u}-t\omega_2}$	$-2\cosh(\tilde{u})\cdot e^{-\omega_2}$	$-e^{+\tilde{u}-(2-t)\omega_2}$
...	...	...	...	...
$E_1^{(2k)}(u)$	$+\frac{1}{4}\frac{\cosh(\frac{\tilde{u}}{2})}{\sinh^3(\frac{\tilde{u}}{2})} + ...$	$+e^{-\tilde{u}-t\omega_2}$	$-2\sinh(\tilde{u})\cdot e^{-\omega_2}$	$-e^{+\tilde{u}-(2-t)\omega_2}$
$E_1^{(2k+1)}(u)$	$-\frac{1}{4}\frac{1}{\sinh^2(\frac{\tilde{u}}{2})} - ...$	$-e^{-\tilde{u}-t\omega_2}$	$-2\cosh(\tilde{u})\cdot e^{-\omega_2}$	$-e^{+\tilde{u}-(2-t)\omega_2}$

(77)

Here we put:

$$\tau = \frac{\omega_2}{\omega_1}, \quad \omega_1 = -i\pi, \quad Im(\omega_2) = 0 \tag{78}$$

Degeneration of $E_1(u)$ up to the second unvanishing order has the form:

$$E_1(u) = \frac{1}{2}\sum_{k=-\infty}^{\infty} \coth(\frac{u}{2} - k\omega_2) \to \begin{cases} 0 < t < 1: & \frac{1}{2} + \frac{1}{2}e^{-\tilde{u}-t\omega_2} \\ t = 1: & \frac{1}{2} - \sinh(\tilde{u})\cdot e^{-\omega_2} \\ 1 < t < 2: & \frac{1}{2} - \frac{1}{2}e^{+\tilde{u}-(2-t)\omega_2} \end{cases} \tag{79}$$

According to this data we will consider the degeneration of $H^{2,2}$ and $H^{2,0}$. First of all let us consider some general remarks. We take into account only the first term in the decomposition $E_1(u)$ because the next terms must vanish in order to avoid the appearance of singularities. So if $0 < t < 2$ we get $E_1(u) = \frac{1}{2}$ after the overscaling. The terms in $H^{2,2}$ containing $E_1^{'}(u)$, containing $(E_1^{'}(u))^2$ in $H^{2,0}$, and $2vg$ in $H^{2,0}$ also vanish up to requirement of the absence of singularities after overscaling $H^{2,4}$. The variables a, b, c, g and z transform in the following way:

$$\tilde{a} = e^{-\chi_a\omega_2}a, \quad \tilde{b} = e^{-\chi_b\omega_2}b, \quad \tilde{c} = e^{-\chi_c\omega_2}c, \quad \tilde{g} = e^{-\chi_g\omega_2}g, \\ z \to \tilde{z} + s\omega_2 \tag{80}$$

There are the following possible relations between t and some parameter s:

$$s = t, 2 - t \tag{81}$$

Let us consider the case of preserving all Lie-Poisson brackets between a, b, c, g after overscaling. The conditions of the absence of singularities in the brackets after overscaling have the following form:

$$\chi_b \geq \chi_g, \quad \chi_c \geq \chi_g, \quad \chi_g \geq \chi_b - \chi_a, \quad \chi_g \geq \chi_c - \chi_a \tag{82}$$

The equality sign corresponds to existance of the Lie-Poisson bracket limit, so:

$$\chi_b = \chi_c = \chi_g, \quad \chi_a = 0 \tag{83}$$

And the brackets take the following form:

$$\begin{gathered}\{v,u\}=1,\quad \{\tilde{c},\tilde{b}\}=0,\quad \{a,\tilde{g}\}=\tilde{c}-\tilde{b},\\ \{\tilde{c},a\}=-2a\tilde{g},\quad \{\tilde{b},a\}=+2a\tilde{g},\quad \{\tilde{b},\tilde{g}\}=-\frac{\tilde{b}\tilde{c}}{a},\quad \{\tilde{c},\tilde{g}\}=\frac{\tilde{b}\tilde{c}}{a}\end{gathered} \tag{84}$$

Casimir operators take the following form:

$$H^{2,4}=\frac{\tilde{b}\tilde{c}}{a}+\tilde{g}^2,\qquad H^{2,3}=\tilde{b}+\tilde{c}, \tag{85}$$

and Hamiltonians in accordance to the value t take the following form:

$$H^{2,2}=(\tilde{c}-\tilde{b})\cdot\frac{1}{2}+\frac{\tilde{b}\tilde{c}}{a}\cdot\frac{1}{2},\qquad 0<t<2$$

$$H^{2,0}=\begin{cases} v^2-\tilde{g}^2\cdot e^{-\tilde{u}}\frac{1}{6}+\frac{\tilde{b}\tilde{c}}{a}\cdot e^{-\tilde{u}}\frac{1}{12}, & 0<t<1\\ v^2-\tilde{g}^2\cdot\cosh(\tilde{u})\frac{1}{3}-(\tilde{b}-\tilde{c})\cdot e^{+\tilde{u}}+\frac{\tilde{b}\tilde{c}}{a}\cdot[\frac{13}{12}e^{+\tilde{u}}+\frac{1}{12}e^{-\tilde{u}}], & t=1\\ v^2-\tilde{g}^2\cdot e^{+\tilde{u}}\frac{1}{6}-(\tilde{b}-\tilde{c})\cdot e^{+\tilde{u}}+\frac{\tilde{b}\tilde{c}}{a}\cdot e^{+\tilde{u}}\frac{13}{12}, & 1<t<2\end{cases} \tag{86}$$

5 Appendix A.

There are main formulas for elliptic functions in this section (they are borrowed from [4] and [9]). First of all we define ϑ-function:

$$\vartheta(z,\tau)=q^{\frac{1}{8}}e^{-\frac{\pi}{4}}(e^{i\pi z}-e^{-i\pi z})\prod_{n=1}^{\infty}(1-q^n)(1-q^ne^{2\pi iz})(1-q^ne^{-2\pi iz}),\quad q=e^{2\pi i\tau} \tag{87}$$

where τ is the complex module. and Eisenstien functions:

$$\begin{gathered}E_1(z,\tau)=\partial_z log\vartheta(z,\tau),\quad E_1(z,\tau)\approx\frac{1}{z}+...\\ E_2(z,\tau)=-\partial_z E_1(z,\tau),\quad E_2(z,\tau)\approx\frac{1}{z^2}+...\\ \frac{E_2'(u)}{E_2(u)-E_2(v)}=E_1(u+v)+E_1(u-v)-2E_1(u)\end{gathered} \tag{88}$$

Their relations to the Weirstrass functions have the following form:

$$\zeta(z,\tau)=E_1(z,\tau)+2\eta_1(\tau)z,\qquad \wp(z,\tau)=E_2(z,\tau)-2\eta_1(\tau). \tag{89}$$

where $\eta_1(\tau)=\zeta(\frac{1}{2})$, and $\zeta(z)$ is Riman ζ-function. Eisenstien functions have the following representations:

$$E_1(z)=\frac{1}{2}\sum_{k=-\infty}^{\infty}\coth(\frac{z}{2}-k\omega_2),\quad E_2(z)=\frac{1}{4}\sum_{k=-\infty}^{\infty}\frac{1}{\sinh^2(\frac{z}{2}-k\omega_2)} \tag{90}$$

$$E_2'(z)=-\frac{1}{4}\sum_{k=-\infty}^{\infty}\frac{\cosh(\frac{z}{2}-k\omega_2)}{\sinh^3(\frac{z}{2}-k\omega_2)} \tag{91}$$

There is the following expression in the matrix of the Lax operator:

$$\varphi(u,z)=\frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}, \tag{92}$$

It has a pole at the point $z=0$ and $res|_{z=0}\varphi(u,z)=1$.

The important identity connected with φ:

$$\varphi(u,v)\varphi(-u,v)=E_2(v)-E_2(u), \quad \varphi'_u(u,v)=\varphi(u,v)(E_1(u+v)-E_1(u)) \tag{93}$$

Behavior on the lattice:

$$\begin{gathered}\theta(z+1)=-\theta(z), \quad \theta(z+\tau)=-q^{-\frac{1}{2}}e^{-2\pi iz}\theta(z),\\ E_1(z+1)=E_1(z), \quad E_1(z+\tau)=E_1(z)-2\pi i,\\ E_2(z+1)=E_2(z), \quad E_2(z+\tau)=E_2(z),\\ \varphi(u+1,z)=\varphi(u,z), \quad \varphi(u+\tau,z)=e^{-2\pi iz}\varphi(u,z).\end{gathered} \tag{94}$$

6 Appendix B.

Let us consider the canonical symplectic form ω on $T^*G\cong\mathcal{G}^*\times G$ ($G\in SL(N,C)$):

$$\omega=\delta<p,g^{-1}\delta g> \tag{95}$$

where $<>$ denotes the Killing form on Lie(G), $p\in sL^*(N,C)$ $g\in SL(N,C)$, δ - external differential operator. We will find the commutation relation between $F(p,g)$ and $H(p,g)$ defined on T^*G. There is Lie-Poisson bracket on T^*G ([11]):

$$\{F,H\}=C_i^{jk}x^i\partial_jF\partial_kH, \tag{96}$$

where C_i^{jk} is the structure constants of the Lie algebra $\mathcal{G}$
$\partial_j=\frac{\partial}{\partial x^j}$ x^i is the coordinate in the space $\mathcal{G}^*$.

We want to rewrite (96) in more convenient form for the next calculations taking into account the explicit dependence of F and H of p and g. Let us represent (96) in the following form:

$$\{F,H\}=X_FH, \tag{97}$$

where X_F is the vector field corresponding to F, so:

$$\{F,H\}=<p,[\frac{\partial F}{\partial p},\frac{\partial H}{\partial p}]>-<g,\{F,H\}>, \tag{98}$$

where $\{F,H\}=\frac{\partial F}{\partial g}\frac{\partial H}{\partial p}-\frac{\partial F}{\partial p}\frac{\partial H}{\partial g}$. Put $H=<p\cdot E_{ji}>$ and $F=<p\cdot E_{lk}>$, so we get the formula (1):

$$\{F,H\}=p_{kj}\delta^{il}-p_{il}\delta^{kj} \tag{99}$$

In the case of $p\in sl(2,C)$ and two point fusion:

$$G=\begin{bmatrix} p_1^0 & p_1^1\\ 0 & p_1^0\end{bmatrix}. \tag{100}$$

Using (98), we get

$$\begin{gathered}\{p_1^{0,ij},p_1^{1,kl}\}=\delta^{il}p_1^{k,j+N}-\delta^{kj}p_1^{i,l+N},\\ \{p_1^{0,ij},p_1^{0,kl}\}=\frac{1}{2}(\delta^{il}p_1^{kj}+\delta^{il}p_1^{k+N,j+N}-\delta^{kj}p_1^{il}-\delta^{kj}p_1^{i+N,l+N}),\end{gathered} \tag{101}$$

7 Appendix C.

Decomposition of $< L^2 >$. Put L in the following form:

$$L = \left[\begin{array}{cc} L^{11} & L^{12} \\ L^{21} & L^{22} \end{array} \right], \tag{102}$$

then:

$$\frac{1}{2} Tr L^2 = (L^{11})^2 + L^{12} L^{21} \tag{103}$$

Let us consider the decomposition of the diagonal and nondiagonal parts of $< L^2 >$ into the Eisenstien functions series in the case of two point.

Diagonal part In this part we will use the property of elliptic functions with equal periods to differ in constant term if they have equal poles with the main parts in the parallelogram of periods.

$$L_{ii} = \sum_i [v_i + (p_a)_{ii} \cdot E_1(z - x_a) + (p_b)_{ii} \cdot E_1(z - x_b)] \tag{104}$$

Let us write the supposed decomposition for TrL_{ii}^2 (we do not write i, supposing summation):

$$\begin{gathered} v^2 + p_a^2 \cdot E_1^2(z - x_a) + p_b^2 \cdot E_1^2(z - x_b) + 2vp_a \cdot E_1(z - x_a) + 2vp_b \cdot E_1(z - x_b) + \\ + 2p_a p_b \cdot E_1(z - x_a) E_1(z - x_b) = \\ = -p_a^2 \cdot E_1^{'}(z - x_a) - p_b^2 \cdot E_1^{'}(z - x_b) + \alpha \cdot E_1(z - x_a) + \beta \cdot E_1(z - x_b) + c_0 + v^2, \end{gathered} \tag{105}$$

where:

$$\alpha = 2p_a p_b \cdot E_1(x_a - x_b) + 2vp_a, \quad \beta = 2p_a p_b \cdot E_1(x_b - x_a) + 2vp_b, \tag{106}$$

Now we must define c_0. In order to do this let us consider $z \to x_a$, representing $z = x_a + \varepsilon$, so as $\varepsilon \to 0$. Expressing c_0 from (84), decomposing all terms in ε series, collecting the similar terms and using the condition $(p_a)_{ii} + (p_b)_{ii} = 0$, we get:

$$c_0 = +p_a p_b \cdot [E_1^{'}(x_a - x_b) + E_1^2(x_a - x_b)] \tag{107}$$

Note that in our calculation c_0 does not depend on the chosen point (on the way of calculation).

Nondiagonal part In this part we will use the property of elliptic function with equal periods to differ only by constant multiplier if they have equal zeros and equal multiplicity of the poles in the parallelogram of periods.

The term of nondiagonal part of $< L^2 >$ in the case of two point is determined in formula (41) of chapter 3. These are the following expressions having θ-functions:

$$\varphi(u, z - x_a)\varphi(-u, z - x_a), \quad \varphi(u, z - x_a)\varphi(-u, z - x_b), \tag{108}$$

where:

$$\varphi(u^{ij}, z - x_s) = \frac{\theta(u^{ij} + z - x_s)\theta^{'}(0)}{\theta(u^{ij})\theta(z - x_s)} \tag{109}$$

and $s = a, b$. Analyzing zeroes and poles of this expressions we will extract the elliptic parts, so this enables us to obtain the decomposition in the Eisenstien functions series depending on $z - x_s$.

Decomposition $\varphi(u, z-x_a)\varphi(-u, z-x_a)$

$$\varphi(u, z-x_a)\varphi(-u, z-x_a) = \frac{\theta(u+z-x_a)\theta(-u+z-x_a)(\theta^{'}(0))^2}{(\theta(u))^2(\theta(z-x_a))^2} \tag{110}$$

Zeros:

$$u \to \pm(z-x_a) \tag{111}$$

Poles:

$$\begin{aligned} u \to 0, \quad \varphi(u, z-x_a)\varphi(-u, z-x_a) \sim -\frac{1}{u^2} \\ z \to x_a, \quad \varphi(u, z-x_a)\varphi(-u, z-x_a) \sim +\frac{1}{(z-x_a)^2} \end{aligned} \tag{112}$$

It follows from the decomposition of θ-functions with respect to small parameter. Thi decomposition takes the following form:

$$\varphi(u, z-x_a)\varphi(-u, z-x_a) = E_1^{'}(u) - E_1^{'}(z-x_a) \tag{113}$$

The similar expression we get for $z-x_b$:

$$\varphi(u, z-x_b)\varphi(-u, z-x_b) = E_1^{'}(u) - E_1^{'}(z-x_b) \tag{114}$$

Decomposition $\varphi(u, z-x_a)\varphi(-u, z-x_b)$

$$\varphi(u, z-x_a)\varphi(-u, z-x_b) = \frac{\theta(u+z-x_a)\theta(-u+z-x_b)(\theta^{'}(0))^2}{(\theta(u))^2\theta(z-x_a)\theta(z-x_b)} \tag{115}$$

Zeros:

$$u \to -(z-x_a), \quad u \to +(z-x_b) \tag{116}$$

Poles:

$$\begin{aligned} u \to 0, \quad \varphi(u, z-x_a)\varphi(-u, z-x_b) \sim -\frac{1}{u^2} \\ z \to x_a, \quad \varphi(u, z-x_a)\varphi(-u, z-x_b) \sim +C\cdot\frac{1}{(z-x_a)} \\ z \to x_b, \quad \varphi(u, z-x_a)\varphi(-u, z-x_b) \sim -C\cdot\frac{1}{(z-x_b)} \end{aligned} \tag{117}$$

Using these zeros and poles we may define the combination of Eisenstien functions:

$$\varphi(u, z-x_a)\varphi(-u, z-x_b) = C\cdot[-E_1(z-x_a) + E_1(z-x_b) - E_1(u) + E_1(u-x_a+x_b)] \tag{118}$$

We may define the coefficient C comparing coefficients in the left and right parts, for example at $z-x_b$:

$$C = \frac{\theta(u+x_b-x_a)\theta^{'}(0)}{\theta(u)\theta(x_a-x_b)} \tag{119}$$

Finally we get:

$$\begin{gathered} \varphi(u, z-x_a)\varphi(-u, z-x_b) = \\ = \frac{\theta(u+x_b-x_a)\theta^{'}(0)}{\theta(u)\theta(x_a-x_b)}\cdot[-E_1(z-x_a) + E_1(z-x_b) - E_1(u) + E_1(u-x_a+x_b)] \end{gathered} \tag{120}$$

and for $\varphi(u, z-x_b)\varphi(-u, z-x_a)$:

$$\begin{gathered} \varphi(u, z-x_b)\varphi(-u, z-x_a) = \\ = \frac{\theta(u+x_a-x_b)\theta^{'}(0)}{\theta(u)\theta(x_b-x_a)}\cdot[-E_1(z-x_b) + E_1(z-x_a) - E_1(u) + E_1(u-x_b+x_a)] \end{gathered} \tag{121}$$

8 Appendix D.

Obtaining Hamiltonians of the new system. Let us find:

$$\frac{1}{2} < L^2 >= (L^{11})^2 + L^{12}L^{21}, \tag{122}$$

in the case of the two point fusion. Let us consider $L^{12}L^{21}$:

$$L^{12}L^{21} =$$
$$= [(p^0)^{12} \cdot \varphi(u, z - x_a) - (p^1)^{12} \cdot \varphi'(u, z - x_a)] \cdot [(p^0)^{21} \cdot \varphi(-u, z - x_a) - (p^1)^{21} \cdot \varphi'(-u, z - x_a)] \tag{123}$$

We need the next formulas to calculate (123):

$$\begin{gathered}
\varphi(+u, z - x_a)\varphi(-u, z - x_a) = E_1'(u) - E_1'(z - x_a) \\
\varphi'(+u, z - x_a)\varphi'(-u, z - x_a) = \\
= -[(E_1'(u))^2 + E_1^2(u)E_1'(u) + E_1(u)E_1''(u) + \frac{1}{3}E_1'''(u)] + \\
+[E_1^2(u) + E_1'(u)] \cdot E_1'(z - x_a) - \frac{1}{6}E_1'''(z - x_a) \\
\varphi(+u, z - x_a)\varphi'(-u, z - x_a) = \\
= -[E_1(u)E_1'(u) + \frac{1}{2}E_1''(u)] + E_1(u) \cdot E_1'(z - x_a) - \frac{1}{2}E_1''(z - x_a) \\
\varphi(-u, z - x_a)\varphi'(+u, z - x_a) = \\
= +[E_1(u)E_1'(u) + \frac{1}{2}E_1''(u)] - E_1(u) \cdot E_1'(z - x_a) + \frac{1}{2}E_1''(z - x_a)
\end{gathered} \tag{124}$$

Put them in (123). Having done the rearrangement and summing with the decomposition of $(L_{11})^2 + (L_{22})^2$, we get:

$$\frac{1}{2}TrL^2 = E_1'''(z - x_a) \cdot H^{2,4} + E_1''(z - x_a) \cdot H^{2,3} + E_1'(z - x_a) \cdot H^{2,2} + H^{2,0} \tag{125}$$

Where :

$$H^{2,4} = -\frac{1}{6}(p^1)^{12}(p^1)^{21} - \frac{1}{6}(p^1)^{11}(p^1)^{11}, \quad H^{2,3} = \frac{1}{2}[(p^0)^{12}(p^1)^{21} + (p^0)^{21}(p^1)^{12}] \tag{126}$$

$$\begin{gathered}
H^{2,2} = -(p^0)^{12}(p^0)^{21} + E_1(u) \cdot [-(p^0)^{12}(p^1)^{21} + (p^0)^{21}(p^1)^{12}] + \\
+(p^1)^{12}(p^1)^{21} \cdot [E_1'(u) + E_1^2(u)] - 2v(p^1)^{11}
\end{gathered} \tag{127}$$

$$\begin{gathered}
H^{2,0} = v^2 + ((p^1)^{11})^2 \cdot [\frac{1}{6}E_1'''(u) + (E_1'(u))^2] + (p^0)^{12}(p^0)^{21} \cdot E_1'(u) + \\
+[(p^0)^{12}(p^1)^{21} - (p^0)^{21}(p^1)^{12}] \cdot [E_1(u)E_1'(u) + \frac{1}{2}E_1''(u)] - \\
-(p^1)^{12}(p^1)^{21} \cdot [(E_1'(u))^2 + E_1^2(u)E_1'(u) + E_1(u)E_1''(u) + \frac{1}{3}E_1'''(u)]
\end{gathered} \tag{128}$$

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